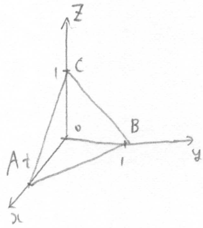
2011

第 1 問

[解]

(1) $\triangle OBC = \frac{1}{2}$, $\triangle OAC = \triangle OAB = \frac{1}{2}t$



$$\triangle ABC = \frac{1}{2}\sqrt{|\vec{AB}|^2|\vec{AC}|^2-(\vec{AB}\cdot\vec{AC})^2}$$
$$=\frac{1}{2}\sqrt{(t^2+1)^2-t^4}$$
$$=\frac{1}{2}\sqrt{2t^2+1}$$

だから、四面体O-ABCの体積を2通りで表して、

$$\frac{1}{3}\cdot\frac{1}{2}\cdot1\cdot1\cdot t=\frac{1}{3}\cdot r\left(\frac{1}{2}+t+\frac{1}{2}\sqrt{2t^2+1}\right)$$
$$t=r\left(1+2t+\sqrt{2t^2+1}\right)$$
$$\therefore r=\frac{t}{1+2t+\sqrt{2t^2+1}} \quad //$$

(2) Pの体積V_1, O-ABCの体積V_2, $f(t)=\frac{V_1}{V_2}$とおく.

$$V_1=\frac{4}{3}\pi r^3$$
$$V_2=\frac{1}{6}t$$

より.

$$f(t)=8\pi\frac{r^3}{t}=8\pi\frac{t^2}{(1+2t+\sqrt{2t^2+1})^3}$$

$g(t)=1+2t+\sqrt{2t^2+1}$ とおくと $g'(t)=2+\frac{4t}{2\sqrt{2t^2+1}}$ で.

$$\frac{f'(t)}{8\pi}=\frac{2t(g(t))^3-t^2\cdot3(g(t))^2\cdot g'(t)}{(g(t))^6}$$
$$=\frac{t}{(g(t))^4}\left[2g(t)-3tg'(t)\right]$$

より. $f'(t)$の符号は〜部($h(t)$とおく)に等しい。

$$h(t)=2(1+2t+\sqrt{2t^2+1})-3t\left(2+\frac{2t}{\sqrt{2t^2+1}}\right)$$
$$=2-2t+2\sqrt{2t^2+1}-\frac{6t^2}{\sqrt{2t^2+1}}$$

$$\frac{1}{2}\sqrt{2t^2+1}\cdot h(t)=(1-t)\sqrt{2t^2+1}+(1-t^2)$$
$$=(1-t)(1+t+\sqrt{2t^2+1})$$

より. 下表をうる

t	0		1	
f'		+	0	−
f		↗		↘

$g(1)=3+\sqrt{3}$

よって$t=1$でmaxをとる.

$$f(1)=\frac{8\pi}{(3+\sqrt{3})^3}=\frac{18-10\sqrt{3}}{9}\pi \quad //$$

$\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}t\\0\\0\end{pmatrix}-\begin{pmatrix}0\\-1\\1\end{pmatrix}$

$\begin{pmatrix}-t\\+1\\0\end{pmatrix}\begin{pmatrix}-t\\0\\1\end{pmatrix}$

$\sqrt{2}\ \sqrt{2}\ \sqrt{2}$

$\frac{\sqrt{3}}{54}-2$

$81-25\cdot3$

$\frac{4}{3}\ \frac{9-5\sqrt{3}}{6}$

$+4t-6t$ $\quad \frac{15-10}{9}$

$1+t$

$3\sqrt{3}(1+\sqrt{3})^3$

$3\sqrt{3}(1+3\sqrt{3}+9+3\sqrt{3})$

$27+\ 3\sqrt{3}(10+6\sqrt{3})$

$6\sqrt{3}(5+3\sqrt{3})$

$\frac{4\pi}{3}\ \frac{1}{9+5\sqrt{3}}\quad 6(9+5\sqrt{3})$

2011

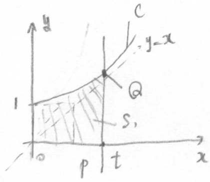
第 2 問

[解] $s=\sin\theta$, $c=\cos\theta$, $x=\tan\theta$ とおく. $0\le\theta<\pi/2$ で $x=\tan\theta$ は非負だから

$$x^2=\frac{1-c^2}{c^2}=1/c^2-1=y^2-1$$

$$\therefore\ x^2-y^2=-1$$

$0\le\theta<\pi/2$ から $0<x$, $1\le y$ で 概形は下図



(1) $S_1=\int_0^t \sqrt{x^2+1}\,dx$

$$=\frac{1}{2}\Big[x\sqrt{x^2+1}+\log\big(x+\sqrt{x^2+1}\big)\Big]_0^t$$

$$=\frac{1}{2}\Big[t\sqrt{t^2+1}+\log\big(t+\sqrt{t^2+1}\big)\Big]_{\#}$$

$$S_2=\frac{1}{2}t\sqrt{t^2+1}_{\#}$$

(2) $S_1-S_2=\frac{1}{2}\log\big(t+\sqrt{t^2+1}\big)$ だから

$$(与式)=\frac{1}{2}\frac{\log\big(t+\sqrt{t^2+1}\big)}{\log t}$$

$$=\frac{1}{2}\frac{\log t+\log\big(1+\sqrt{1+1/t^2}\big)}{\log t}$$

$$\longrightarrow \frac{1}{2}_{\#}\quad (t\to\infty)$$

$x=\frac{s}{c}$

$x^2=\frac{1-c^2}{c^2}=\frac{1}{c^2}-1=y^2-1$

c^{-1}

$-\frac{s}{c}$

$y=\frac{1}{c}$ $\quad c=\frac{1}{y}$

$\frac{dx}{d\theta}=\frac{1}{c^2}$

p $\quad x=\tan\theta$ $\quad \frac{}{c^2}$

$\frac{1}{c^2}$ $\quad \frac{d\theta}{c^2}$